EEL 711/ELL 711 Minor Test II Semester I 2015-2016

Answer all questions (Marks: Q.1: 24, Q.2: 16) Full Marks: 40

1. A real-valued Gaussian random vector $\underline{X} = [X_1\ X_2\ X_3]^T$ has a $\mathcal{N}\left(\underline{\mu}, \underline{K}\right)$ distribution, such that

$$\underline{\mu} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \underline{K}^{-1} = \frac{1}{(1-\rho^2)} \begin{bmatrix} 1 & -\rho & 0 \\ -\rho & (1+\rho^2) & -\rho \\ 0 & -\rho & 1 \end{bmatrix}, \quad \det\left(\underline{K}^{-1}\right) = \frac{1}{(1-\rho^2)^2}, \quad -1 < \rho < 1.$$

(a) Let $Y_i = X_i - 4$, $i = 1, 2, 3$. If the joint p.d.f. of Y_1, Y_2, Y_3 is given by

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \frac{1}{C} \exp\left\{ -\frac{[y_1^2 + (1+\rho^2)\, y_2^2 + y_3^2 + 8 + A(y_1y_2 + y_2y_3) + B(y_1 - y_3)]}{2\,(1-\rho^2)} \right\},$$

then find A, B, C. [6]

(b) Find $\underline{K}$. [4]

(c) Let $V_1 = 3X_1 - 6$, $V_2 = 2X_2 - 8$, $V_3 = X_3 - 6$, and $\rho = \sqrt{0.6}$.

i. Calculate $\mathbf{E}\left\{ \begin{bmatrix} V_1 \\ V_3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix}^T \right\}$ and $\mathbf{E}\left[V_1^2 V_2 V_3\right]$. [6]

ii. Let $U_1 = a_{11}V_1$, $U_2 = a_{21}V_1 + a_{22}V_3$, such that $a_{11}, a_{22} > 0$, and U_1 and U_2 are i.i.d. $\mathcal{N}(0, 1)$ random variables. Calculate a_{11}, a_{21}, a_{22}. [8]

2. A communication receiver with 2 antennas receives a complex-valued transmitted signal sample $e^{j\phi}$, where $\phi \in [-\pi, \pi)$, through in-phase and quadrature channels. In the in-phase and quadrature channels, the 2×1 received signal vectors are $\underline{X}$ and $\underline{Y}$, such that

$$\underline{Z} = \underline{X} + j\underline{Y} = e^{j\phi}(\underline{G} + j\underline{H}) + \underline{N}_Z\,,$$

where $\underline{G}, \underline{H}$ are random channel gains and $\underline{N}_Z$ is the additive noise, such that

$$\underline{G}, \underline{H} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Omega \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right), \quad \mathbf{E}\left[\underline{G}\underline{H}^T\right] = \Omega \begin{bmatrix} 0 & -\rho \\ 1-\rho & 0 \end{bmatrix},$$

$$\underline{N}_Z \sim \mathcal{CN}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 2\sigma^2 \underline{I}_2 \right), \quad \Omega, \sigma > 0, \quad 0 < \rho < 1,$$

and $\underline{N}_Z$ is independent of $\underline{G}$ and $\underline{H}$.

(a) Find $\underline{K}_Z$, the covariance matrix of $\underline{Z}$. [4]

(b) For what value of ρ is $\underline{Z}$ circular? [4]

(c) Let $\underline{U} = [U_1\ U_2]^T = \underline{X} + \underline{Y}$, $\underline{V} = [V_1\ V_2]^T = \underline{X} - \underline{Y}$, and $\underline{W} = [W_1\ W_2]^T = \underline{U} + j\underline{V}$. For the value of ρ in (b), find the c.f.s $\Psi_{W_1,W_2}(j\nu_1, j\nu_2)$ and $\Psi_{U_1,V_2}(j\omega_1, j\omega_2)$. [4+4]